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Application of Wavelet Transform and its Advantages Compared to Fourier Transform

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ABSTRACT

Wavelet analysis is an exciting new method for solving difficult problems in mathematics, physics, and engineering, with modern applications as diverse as wave propagation, data compression, signal processing, image processing, pattern recognition, computer graphics, the detection of aircraft and submarines and other medical image technology. Wavelets allow complex information such as music, speech, images and patterns to be decomposed into elementary forms at different positions and scales and subsequently reconstructed with high precision. Signal transmission is based on transmission of a series of numbers. The series representation of a function is important in all types of signal transmission. The wavelet representation of a function is a new technique. Wavelet transform of a function is the improved version of Fourier transform. Fourier transform is a powerful tool for analyzing the components of a stationary signal. But it is failed for analyzing the non stationary signal where as wavelet transform allows the components of a non-stationary signal to be analyzed. In this paper, our main goal is to find out the advantages of wavelet transform compared to Fourier transform.

Keywords: Fourier transform, wavelet, wavelet transform, time-frequency signal analysis

1. Introduction

In 1982 Jean Morlet a French geophysicist, introduced the concept of a `wavelet'. The wavelet means small wave and the study of wavelet transform is a new tool for seismic signal analysis. Immediately, Alex Grossmann theoretical physicists studied inverse formula for the wavelet transform. The joint collaboration of Morlet and Grossmann [5] yielded a detailed mathematical study of the continuous wavelet transforms and their various applications, of course without the realization that similar results had already been obtained in 1950's by Calderon, Littlewood, Paley and Franklin. However, the rediscovery of the old concepts provided a new method for decomposing a function or a signal. For details one can see Morlet *et al*. [8], Debnath [4].

Wavelet analysis is originally introduced in order to improve seismic signal analysis by switching from shortime Fourier analysis to new better algorithms to detect and analyze abrupt changes in signals Daubechies [2,3], Mallat [6]. In time-frequency analysis of a signal, the classical Fourier transform analysis is inadequate because Fourier transform of a signal does not contain any local information. This is the major drawback of the Fourier transform. To overcome this drawback, Dennis Gabor in 1946, first ntruduced the windowed-Fourier transform, i.e. short-time Fourier transform known later as Gabor transform. Meyer [7] found the existing literature of wavelets. Later many eminent mathematicians e.g. I. Daubechies, A. Grossmann, S. Mallat, Y. Meyer, R. A. deVore, Coifman, V. Wickerhauser made a remarkable contribution to the wavelet theory. The modern applications of wavelet theory as diverse as wave propagation, data compression, signal processing, image processing, pattern recognition, computer graphics, the detection of aircraft and submarines, improvement of CAT scans and some other medical image technology etc. In this study, our main goal is to find out the advantages of wavelet transform compared to Fourier transform.

2. Wavelet

Wavelet may be seen as a complement to classical Fourier decomposition method. Suppose, a certain class of functions is given and we want to find 'simple functions' $f_0, f_1, f_3, \ldots\ldots$ such that each

$$f(x) = \sum_{n=0}^{\infty} a_n f_n(x) \tag{1}$$

for some coefficients a_n.

Wavelet is a mathematical tool leading to representations of the type (1) for a large class of functions f.

Wavelet theory is very new (about 25 years old) but has already proved useful in many contexts.

Definition (Wavelet)

A wavelet means a small wave (the sinusoids used in Fourier analysis are big waves) and in brief, a wavelet is an oscillation that decays quickly.

Equivalent mathematical conditions for wavelet are :

$$\int_{-\infty}^{\infty} |\psi(t)|^2 \, dt < \infty ; \tag{2}$$

$$\int_{-\infty}^{\infty} |\psi(t)| dt = 0 ; \tag{3}$$

$$\int_{-\infty}^{\infty} \frac{\left|\widehat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty ; \tag{4}$$

where $\widehat{\psi}(\omega)$ is the Fourier Transform of $\psi(t)$. Equation (4) is called the admissibility condition.

3. Wavelet Transform

Jean Morlet in 1982, introduced the idea of the wavelet transform and provided a new mathematical tool for seismic wave analysis. Morlet first considered wavelets as a family of functions constructed from translations and dilations of a single function called the "mother wavelet" ψ(t). They are defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right), \qquad a,b \in \mathrm{R},\ a \neq 0 \tag{5}$$

The parameter a is the scaling parameter or scale, and it measures the degree of compression. The parameter b is the translation parameter which determines the time location of the wavelet. If $|a| < 1$, then the wavelet in (5) is the compressed version (smaller support in time- domain) of the mother wavelet and corresponds mainly to higher frequencies. On the other hand, when $|a| > 1$, then ψa,b(t) has a larger time-width than $\psi(t)$ and corresponds to lower frequencies. Thus, wavelets have time-widths adapted to their frequencies. This is the main reason for the success of the Morlet wavelets in signal processing and time-frequency signal analysis.

4. Wavelet Series and Wavelet Coefficients

If a function $f \in L_2(\mathrm{R})$, the series

$$\sum_{j \in z} \sum_{k \in Z} \langle f, \psi_{j,k} \rangle \ \psi_{j,k}(t) \tag{6}$$

is called the wavelet series of f and

$$\langle f, \psi_{j,k} \rangle = d_{j,k} = \int_{-\infty}^{\infty} f(t) \psi_{j,k}(t) dt \tag{7}$$

is called the wavelet coefficients of f.

5. Signal

A signal is given as a function f which has a series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then all information about the function f is stored in the coefficients $\{a_n\}_{n=0}^{\infty}$.

6. Classification of Signals

We can split the class of signals into two classes, namely:

- Continuous signals
- Discrete signals

Continuous Signals:

The signals which is described by a continuous function (for example a recording of a speech signal or music signal, which measures the current in the cable to the loudspeaker as a function of time) is called continuous signals Christensen [1], Mallat [6].

Here is a concrete example concerning sound signals:

An Example

Wavelets are frequently used to remove noise from music recordings. The main idea is to think about a music signal as consisting of the music itself to which some noise is added.

The music signal itself describes how the music changes in time; we can think about the signal as the current through the loudspeaker when we play a recording. Expanding the music piece via wavelets means that we represent this signal via the coefficients $\{d_{j,k}\}$ in (7); the coefficients "tell when something happens in the music". The noise contribution is usually small compared to the music, but irritating for the ears; it also contributes to the coefficients $\{d_{j,k}\}$, but usually less than the music itself. The idea is now to remove the coefficients in (7) which are smaller than a certain threshold value (strictly speaking, this procedure is not applied on the signal itself, but on its so called wavelet transform). To be more precise, this means that these coefficients are replaced by zeroes; thus, one lets the remaining coefficients represent the music. The idea is to remove the part of the signal which has no relationship to the music; unfortunately, the above procedure also might cancel smaller parts of the music itself, but still the result is usually considered by the ears as an improvement Christensen [1], Mallat [6].

Discrete Signals:

The signals which is described by a sequence of numbers or pairs of numbers is called discrete signals Christensen [1], Mallat [6].

Here mentioned one example of discrete signals:

An Example

A digital black-white photo consists of a splitting of the picture into a large number of small squares, called pixels; to each pixel, the camera associates a light intensity, measured on a scale from, say, 0 (completely white) to 256 (completely black). Put together, this information constitutes the picture. Thus, mathematically a photo consists of a sequence of pairs of numbers, namely, a numbering of the pixels together with the associated light intensity Christensen [1], Mallat [6].

7. Some Application of Wavelets

Wavelets are a powerful statistical tool which can be used for a wide range of applications, namely

- Signal processing
- Data compression
- Smoothing and image denoising
- Fingerprint verification
- Biology for cell membrane recognition, to distinguish the normal from the pathological membranes
- DNA analysis, protein analysis
- Blood-pressure, heart-rate and ECG analyses
- Finance (which is more surprising), for detecting the properties of quick variation of values
- In Internet traffic description, for designing the services size
- Industrial supervision of gear-wheel
- Speech recognition
- Computer graphics and multifractal analysis
- Many areas of physics have seen this paradigm shift, including molecular dynamics, astrophysics, optics, turbulence and quantum mechanics.

Wavelets have been used successfully in other areas of geophysical study. Orthonormal wavelets, for instance, have been applied to the study of atmospheric layer turbulence. In one study by J.F. Howell and L. Mahrt, turbulence measurements were taken over a nine-hour period and analyzed using wavelet decomposition. In another study by Brunet and Collineau, turbulence data recorded over a corn crop was analyzed using the wavelet transform.

Wavelets have also been used to analyze seafloor bathymetry or the topography of the ocean floor. In one study by Sarah Little, the use of wavelet analysis revealed patterns, trends, and structures that may be overlooked in raw data.

Also, the use of methods like local oracles allowed for separation of data in regions of interest.

Several other geophysical applications such as analysis of marine seismic data and characterization of hydraulic conductivity distributions have also been used. The usefulness of wavelets in data analysis is clear, particularly in the field of geophysics, where large and cumbersome data sets abound. Studies such as the atmospheric layer turbulence and corn crop turbulence have further shown the proficiency of wavelets in the analysis of time-dependent data sets Christensen [1], Debnath [4], Meyer [7].

8. Data Compression

Consider the following sequence of numbers (which represent a given signal)

56	40	8	24	48	48	40	16

To these eight numbers we now associate eight new numbers, which appear in the following way. First, we consider the above eight numbers as a series of four pairs of numbers, each containing two numbers. We now replace each of these pairs of numbers by two new numbers, namely, their average and the difference between the first number in the pair and the calculated average. The first pair of numbers in the given signal consists of the numbers 56 and 40; our procedure replaces then by the new pair consisting of the numbers $\frac{56+40}{2}=48$, 56-48=8. Applying this procedure on all four pairs we obtain the following sequence of numbers:

48	8	16	-8	48	0	28	12

In mathematical terms, we have replaced the pair (a,b) by a new pair (c,d) given by

$$c=\frac{a+b}{2}\ ,\ d=a-\frac{a+b}{2}=\frac{a-b}{2}. \tag{8}$$

Let us write the new sequence of numbers under the original sequence:

56	40	8	24	48	48	40	16
48	8	16	-8	48	0	28	12

In some sense, the obtained sequence in the second row contains the same information as the original sequence in the first row: we can always come from the second row and back to the first by inverting the above procedure. That is, we have to solve the equations (8) with respect to a,b and apply the obtained formula to the four pairs of numbers in the second row. Solving the equations (8) with respect to a,b gives

$$a=c+d,\ b=c-d \tag{9}$$

So the inverse transform is exactly as easy to apply as the transform itself. When we go back to the original signal from the second row we speak about reconstruction of the original information. If our purpose is to store the information or to send it, we can equally well work with the second row as with the first: we just have to remember to transform back at a certain stage. However, at this moment it is not clear that we gain anything by the transformation. We keep this question open for a moment, and apply the procedure once more; but only on the numbers appearing as averages. In other words, we let the numbers 8, -8, 0, 12 (Calculated as differences) stay in the table, and repeat the process on the numbers 48, 16, 48, 28 i.e. on the pairs 48, 16 and 48, 28. That is, we calculate average and difference (in the above sense) for each of these pairs.This leads to the numbers 32, 16, 38, 10 which are placed on the free places in the table; this gives us the table

32	8	16	-8	38	0	10	12

These eight numbers also represent the original information: we just have to invert the above procedure twice, then we are back at the original sequence. However, some bookkeeping is involved: we have to keep track of which numbers we calculated as averages, and which were differences.

Finally, we repeat the process on the numbers 32, 16, 38, 10. The numbers 16, 10 appeared as differences, so we do not change them; the numbers 32, 38 appeared as averages, so we replace them by their average 35 and the difference 32- 35= -3. Thus, we obtain the table

35	8	16	-8	-3	0	10	12

So, far we have only argued that the performed operations do not change the information, in the sense that we can come back to the original numbers by repeated application of the inversion formula (9). In order to show that we have gained something, we now apply thresholding on the numbers in the final table. In popular words, this means that we remove the numbers which numerically are smaller than a certain fixed number ; more precisely, we replace them by zeros. If we for example decide to remove all numbers which numerically are smaller than 4, we obtain the table

32	8	16	-8	0	0	10	12

Let us now perform the reconstruction, i.e. apply the inversion formula (9), based on these numbers ; note that we have to perform the inversion in three steps, paying close attention to which numbers we obtained as averages and differences in each step. Then we obtain the sequence in the first row below; in order to compare with the original sequence we place the signal we started with in the second row:

59	43	11	27	45	45	37	13
56	40	8	24	48	48	40	16

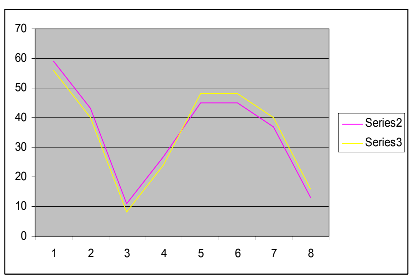

Fig.1. The original signal and reconstruction after thresholding with 4.

We notice that the numbers in the first row are quite close to the original sequence. If we are rough and use thresholding by 9 in the table obtained after three steps of calculating averages and differences, we obtain the table

35	0	16	0	0	0	10	12

Here, reconstruction yields (we again repeat the original sequence in the second row)

51	56	19	19	45	45	37	13
56	40	8	24	48	48	40	16

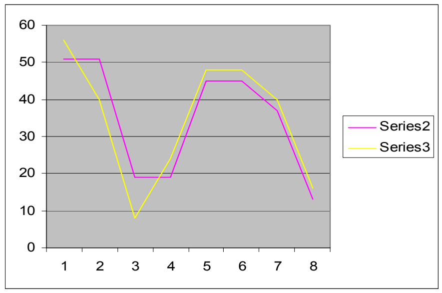

Fig.2. The original signal and reconstruction after thresholding with 9.

Observe that even though the rough thresholding is worse, the reconstructed signal still follows the shape of the original signal quite well, except in the neighborhood of points where the signal oscillates heavily Christensen [1].

9. Storing Fingerprint Electronically Using Wavelet

We now discuss how the FBI (Federal Bureau of Investigation) in the USA uses wavelets as a tool to store fingerprints electronically.

For many years, the FBI stored their fingerprints in paper format in a highly secured building in Washington; they filled an area which had the same size as football field. If one needed to compare a fingerprint in San Francisco with the stored fingerprints, one had to mail it to Washington. Furthermore, comparison of the fingerprints was done manually, so it was a quite slow process. For these reasons, the FBI started to search for ways to store the fingerprints electronically; this would facilitate transmission of the information and the search in the archive.

We can consider a fingerprint as a small picture, so a natural idea is to split each square-inch into, say, 256×256 pixels, to which we associate a grey-tone on a scale from for example 0 (completely white) to 256 (completely black). This way we have kept the essential information in the form of a sequence of pairs of numbers, namely, the pairs consisting of a numbering of the pixels and the associated grey-tone.

This sequence can easily be stored and transmitted electronically, i.e. it can if necessary be sent rapidly to San Francisco and be compared with a given fresh fingerprint.

The remaining problem has to do with the size of the archive. With reasonable accuracy, the above procedure will represent each fingerprint by a sequence of numbers which uses 10 Mb, i.e. information corresponding to about 10 standard diskettes. The FBI has more than 30 million sets of fingerprints (each consisting of 10 fingers) and receives about 30000 new fingerprints each day. Thus, we are speaking about enormous sets of data , and it is necessary to do some compression in order to be able to handle them. This has to be done in a way such that the structure of the fingerprints is kept; see the discussion about data compression in section 8.

This is where wavelets enter the scene. The FBI started to search for efficient ways to compress data and became interested in wavelets. It had been known for some time that wavelets are good to represent many types of signal using few coefficients and the end of the story was that the FBI decided to use a variant of the wavelets discussed here. In fact, the FBI uses Daubechies' biorthogonal spline wavelets; here ψ is piecewise polynomial and $f(t) = \sum_{j \in Z} \sum_{k \in Z} d_{j,k} \psi_{j,k}(t)$ is satisfied with coefficients which are a slight modification of (7). In the concrete case it turned out that it was enough to represent a fingerprint using about 8% of the original information, which makes it possible to store it on a single diskette. The compression method has the impressive name *The Wavelet Scalar Quantization Gray- scale Fingerprint Image Compression Algorithm*, Usually abbreviated WSQ.
The fundamental example data compression in section 8 gives part of the explanation why such an efficient compression is possible. A fingerprint (or almost any other picture) has a certain structure, which implies that the sequence of gray – tones associated with the pixels are not random numbers. Consider for example a picture showing a house: here the gray -tones will be almost constant i.e. the walls and only around corners and windows will there be large variations as described in section 8, large areas with almost constant gray-tones (and therefore small differences) give good options for compression and in the concrete example with fingerprints, excellent results are obtained. See Fig.3 and Fig.4 , which show an original fingerprint and its compressed version, respectively. The original fingerprint exists electronically and can be downloaded from the NIST (National Institute of Standards and Technology) [13(a)].

Fig.3. Original fingerprint.

Fig.4. Fingerprint, compressed using wavelets and reconstructed.

The compressed fingerprint is made using software developed by the mathematician

Mladen Victor Wickerhauser. The software has the name ID#55291 is owned by the company Wickerhauser Consulting and can be downloaded from Wickerhauser's home page at Washinngton University in St. Louis, USA [13(b)].
under "my free software".

10. Fingerprint Verification

Fingerprint verification is one of the most reliable personal identification methods and it plays a very important role in forensic and civilian applications. However, manual fingerprint verification is so tedious, time-consuming and expensive in that it is incapable of meeting today's increasing performance requirements. Hence, an automatic fingerprint identification system (AFIS) is widely needed. Here we mentioned one real example of Fingerprint verification:

In Singapore, a new security system was introduced in Hitachi Tower (a 37-storey office building) in 2003 : now, the 1500 employees get access to the building by scanning their fingers. The scanner uses infrared rays to trace the hemoglobin in blood in order to capture the vein patterns in the finger; these patterns determine the person uniquely. After comparing with the scanned data in an electronic archive, it is decided whether the person can get in or not Christensen [1], Meyer [7].

11. Comparison Wavelet Transform with Fourier Transform

The wavelet transform is often compared with the Fourier transform. Fourier transform is a powerful tool for analyzing the components of a stationary signal (a stationary signal is a signal where there is no change in the properties of signal). For example, the Fourier transform is a powerful tool for processing signals that are composed of some combination of sine and cosine signals (sinusoids) Mallat [6].

The Fourier transform is less useful in analyzing non-stationary signal (a non-stationary signal is a signal where there is change in the properties of signal). Wavelet transforms allow the components of a non-stationary signal to be analyzed. Wavelets also allow filters to be constructed for stationary and non-stationary signals Wells [11], Strang [9].

The Fourier transform shows up in a remarkable number of areas outside of classic signal processing. Even taking this into account, we think that it is safe to say that the mathematics of wavelets is much larger than that of the Fourier transform. In fact, the mathematics of wavelets encompasses the Fourier transform. The size of wavelet theory is matched by the size of the application area. Initial wavelet applications involved signal processing and filtering. However, wavelets have been applied in many other areas including non-linear regression and compression. An offshoot of wavelet compression allows the amount of determinism in a time series to be estimated Wojtaszczyk [12].

The main difference is that wavelets are well localized in both time and frequency domain whereas the standard Fourier transform is only localized in frequency domain. The Short-time Fourier transform (STFT) is also time and frequency localized but there are issues with the frequency time resolution and wavelets often give a better signal representation using Multiresolution analysis Walnut [10].

Fourier transform is based on a single function $\psi(t)$ and that this function is scaled. But for the wavelet transform we can also shift the function, thus generating a two-parameter family of functions $\psi_{a,b}(t)$ defined by (5) Debnath [4].

12. Why Do We Analyze Wavelet?

All types of signal transmission are based on transmission of a series of numbers. For signal transmission or signal storage the first step is to convert the given information to a series of numbers. To do this we need to represent a function f as a series representation. The function f is stored in the coefficients of the series and we can send only the coefficients. In practice we cannot send an infinite sequence of numbers. It is possible to send only a finite sequence of numbers. For good approximation usually this number forces to be large.

For series representation of a function, we consider a given function or signal f as

$$f(x) = \sum_{0}^{\infty} a_n f_n(x) \tag{10}$$

Where a_n's are constant coefficients and $f_0, f_1, f_2, \ldots$ are simple functions.

In signal analysis it is common to consider a function in L_2 (R).

i.e. $L_2(\mathbf{R}) = \left\{ \left\{ f : \mathbf{R} \to \mathbf{C} / \int_R |f(x)|^2 < \infty \right\} \right.$. This is never periodic except $f = 0$.

In that case we consider a wavelet function ψ such that

$$f(x) = \sum_{j\in Z} \sum_{k\in Z} d_{j,k} \psi_{j,k}(x)$$

where $d_{j,k}$ are wavelet coefficients and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ are the translated and scaled version of wavelet ψ.

For a periodic function the classical method is Fourier transform. But the main drawback of Fourier transform is that we lose our time information which is very important. The wavelet tool is the new tool to represent a function like (10). In the wavelet transform we do not lose the time information, which is useful in many contexts.

Some Advantages of Wavelet Theory:

a) One of the main advantages of wavelets is that they offer a simultaneous localization in time and frequency domain.
b) The second main advantage of wavelets is that, using fast wavelet transform, it is computationally very fast.
c) Wavelets have the great advantage of being able to separate the fine details in a signal. Very small wavelets can be used to isolate very fine details in a signal, while very large wavelets can identify coarse details.
d) A wavelet transform can be used to decompose a signal into component wavelets.

e) In wavelet theory, it is often possible to obtain a good approximation of the given function f by using only a few coefficients which is the great achievement in compare to Fourier transform.
f) Most of the wavelet coefficients $\{d_{j,k}\}_{|j|,|k|\geq N}$ vanish for large N.
g) Wavelet theory is capable of revealing aspects of data that other signal analysis techniques miss the aspects like trends, breakdown points, and discontinuities in higher derivatives and self-similarity.
h) It can often compress or de-noise a signal without appreciable degradation.

13. Results and Discussion

Fig.1 shows that the original signal and reconstruction signal after thresholding with 4. From Fig.1, we observe that the reconstructed signal are quite close to the original signal. Fig.2 shows that the original signal and reconstruction signal after thresholding with 9. From Fig.2, we observe that even though the rough thresholding is worse, the reconstructed signal still follows the shape of the original signal quite well, except in the neighborhood of points where the signal oscillates heavily. Hence we can say that data compression is the great achievement of wavelet transform i.e. in wavelet transform, it is often possible to obtain a good approximation of the given signal f by using only a few coefficients. Fig.3 shows the original fingerprint and Fig.4 shows the compressed fingerprint using wavelets and reconstructed. However, manual fingerprint verification is so tedious, time-consuming and expensive in that it is incapable of meeting today's increasing performance requirements. Hence, an automatic fingerprint identification system (AFIS) is widely needed. Wavelet transform which has wide range of applications such as image compression is used in this paper for Fingerprint verification. It describes the design and implementation of an off-line fingerprint verification system using wavelet transform. In this method, matching can be done between the input image and the stored template without resorting to exhaustive search using the extracted feature. The experimental results show that the wavelet transform based approach is better than the existing minutiae based method and it takes less response time which is more suitable for on-line verification with high accuracy. Let us finally mention that compression of fingerprints in principle also can be performed using Fourier transform. However, this classical method is less efficient: at the rate of compression The FBI uses for the wavelet method it would no longer be possible to follow the contours in a reconstructed fingerprint and the result would be useless in this special context. Fourier analysis is a mathematical technique for transforming our view of the signal from time-based to frequency-based. In transforming to the frequency domain, time information is lost which is very important. This is the main drawback of Fourier transform. In the wavelet transform we do not lose the time information which is useful in many contexts.

In Fourier analysis signal properties do not change over time. This drawback is not very important. But most interesting signals contain numerous non- stationary or transitory characteristics like drift, trends, abrupt changes and beginnings and ends of event. These characteristics are often the most important part of the signal. The classical Fourier analysis is not suited for detecting them but wavelet analysis is suited for detecting them.

14. Conclussion

In this paper, we have discussed about some applications of wavelet such as data compression, recording of a sound signal, music signal, fingerprint verification with the help of a wavelet transform. We have also tried to comparative discussion of Fourier transform and wavelet transform mentioning the drawback of Fourier transform, besides this we have discussed the advantages of wavelet transform. From our above discussion it is clear that the experimental results show that the wavelet transform based approach is better than the existing minutiae based method and it takes less response time which is more suitable for online verification with high accuracy. Finally, we can say that wavelet transform is a reliable and better technique than that of Fourier transform technique.

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